Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Turing instability in the Lengyel–Epstein fractional Laplacian system

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Abstract

This paper discusses a space-fractional version of the conventional Lengyel–Epstein CIMA reaction model. First, we prove the global existence, uniqueness, and boundedness of a unique solution. We next investigate the system fundamental analytic properties. Following this, we establish the conditions on the reactor size and diffusion coefficient such that the system does not allow positive steady-state solutions that are not constant. Finally, the stability of constant steady-state solutions for ODE and PDE models is studied.

Keywords: Turing instability; Lengyel–Epstein; Regional fractional Laplacian; Nonconstant positive solutions

1 Introduction

Trembley [1] discovered the regeneration phenomenon of hydra in 1744, making it one of the earliest and best recognized instances of morphogenesis. To model phenomena and important forms in the formation of biological patterns. A. Turing introduced the concept of "diffusion-driven instability" in 1952. Specifically, Turing [2] contended that given a system including two interacting substances, varying diffusion speeds may result in heterogeneous distributions of these reactants. This concept was subsequently and effectively advanced theoretically by Gierer and Meinhardt [3] in their modeling of hydra regeneration, and it was further mathematically validated to demonstrate the ability to generate spikes and peaks (see, e.g., [4]). Turing's concept has garnered significant interest from several researchers and has been effectively advanced on theoretical foundations. It has been examined not just in biological and chemical domains but also in economics, semiconductor physics, and stellar creation (see [5]). Nonetheless, the quest for Turing patterns in actual chemical or biological systems proved to be challenging. In the early 1990s, De Kepper et al. [6] discovered the formation of stationary three-dimensional (though nearly two-dimensional) structures while working with the chlorite-iodide-malonate acid (CIMA) reaction. This represented the first experimental evidence of Turing patterns, nearly 40 years after the publication of [2]. The presence of five reactants in the CIMA reaction complicates the mathematical analysis of the system. Lengyel and Epstein [7, 8] successfully reduced the original system to a two-dimensional framework, known as the Lengyel–

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Epstein model. A comprehensive historical overview of the evolution of the CIMA reaction model and associated experiments is available in Epstein and Pojman [5]. The following three chemical reaction schemes elucidate the CIMA reaction:

$$\begin{cases} MA + I_2 \to IMA + I^- + H^+, \\ ClO_2 + I^- \to \frac{1}{2} I_2 + ClO_2^-, \\ ClO_2^- + 4I + 4H^+ \to Cl^- + 2I_2 + 2H_2O. \end{cases}$$

By examining the empirical rate laws associated with these processes and disregarding constant factors the reaction model was simplified to the standard Lengyel–Epstein model utilizing two independent variables U and V, which correspond to the iodide concentration (I^-) and the chlorite concentration (ClO_2). This model is derived from two differential equations. The rates of change in changing concentrations must be equilibrated by studies, and it is challenging for the reactions to be converted into differential equations. Let U and V represent the concentrations of iodide and chlorite, respectively. Then the suggested rate equations are articulated as

$$\begin{cases} \frac{dU}{d\tau} = \frac{k_1[MA][I_2]}{k_2[I_2]} - k_4[ClO_2]U - 4\left(k_5[H^+]UV + k_6[I_2]\dfrac{UV}{\ell + U^2}\right), \\ \frac{dV}{d\tau} = k_4[ClO_2]U - \left(k_5[H^+]UV + k_6[I_2]\dfrac{UV}{\ell + U^2}\right), \end{cases}$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ represent the rate constants, and ℓ is also a constant. The rate conditions are linked to all particles, including those involved with the method [9]. However, it may be suggested during research that the iodide and chlorite concentrations must fluctuate more rapidly than other components. In this regard, it is reasonable to presume that efficacy remains consistent with the aggregation of various molecules as long as the reaction is maintained. This assumption need not be used thereafter. Nonetheless, we must choose to engage with authentic values in any mathematical models. Following the selection of the appropriate path for sustainability, the k_5 term will be disregarded, as noted by Lengyel and Epstein (1991), for no apparent reason other than the certainty that the hydrogen concentration H^+, being the lightest nucleus, is minimal. The aforementioned theories lead to the following system of differential equations:

$$\begin{cases} \frac{dU}{d\tau} = m - nU - 4p\left(\dfrac{UV}{\ell + U^2}\right), \\ \frac{dV}{d\tau} = pU - p\left(\dfrac{UV}{\ell + U^2}\right), \end{cases}$$

where $m, n, p > 0$. According to Jang et al. [10], the Lengyel–Epstein model is delineated as follows with the completion of all operations:

$$\begin{cases} \frac{du}{dt} = \Delta u + a - u - \dfrac{4uv}{1 + u^2}, \\ \frac{dv}{dt} = \sigma\left[c\Delta v + b\left(u - \dfrac{uv}{1 + u^2}\right)\right], \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Let $u = u(x,t)$ and $v = v(x,t)$ represent the chemical concentrations of the activator iodide (I^-) and the inhibitor chlorite $\left(ClO_2^-\right)$ at time $t > 0$ and point $x \in \Omega$, respectively. The constants a, c, and

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b are all strictly positive. The Lengyel–Epstein system is the focus of a notable study by Ni and Tang [11], who have shown, by experimental and statistical approaches, that the system cannot sustain nonconstant steady states if the starting reactant concentrations, reactor size, or effective diffusion rate are inadequately large. In 2008, Yi et al. [12] performed research that explicitly delineated the system characteristics required to ensure that the spatial homogeneous equilibrium and the spatial homogeneous periodic solutions are either Turing-unstable or diffusively unstable. This work exemplifies scientific investigation. In 2009, Yi et al. [13] established that the constant equilibrium solution is globally asymptotically stable through the use of a particular Lyapunov functional. Du and Wang [14] introduced various spatially nonhomogeneous periodic solutions, whereas Jin et al. [15] selected different bifurcation parameters for the identical model, akin to [14]. In 2013, Wang and Zhao [16] utilized the Hopf bifurcation theorem to ensure the fundamental conditions for the stability of equilibrium points. They demonstrated their conclusions using numerical examples. In 2014, Lisena [17] investigated the dynamics of the Lengyel–Epstein system and alleviated the restrictions on the global asymptotic stability of the steady-state solution. In 2015, Shoji and Ohta [18] introduced computer simulations of three-dimensional Turing patterns within the framework of the Lengyel–Epstein model. The first endeavor to extend the model was accomplished in [19], demonstrating how alternative approximations of the response term may be examined comprehensively. The same model has been reanalyzed in [20], whereby the authors delineated adequate requirements for the nonexistence of Turing patterns. The time-fractional Lengyel–Epstein system has been studied [21], where the authors determined sufficient conditions for the local and global asymptotic stability of the unique equilibrium. The Lengyel–Epstein and Degn–Harrison models have been studied in [22], where the authors determined sufficient conditions for the global asymptotic stability of the steady-state solution. The following model is used by the authors in [23]:

$$\begin{cases} u_t = \nabla^\gamma u + a - u - \dfrac{4uv}{1+u^2}, \\ v_t = d\nabla^\gamma v + b\left(u - \dfrac{uv}{1+u^2}\right), \end{cases}$$

where ∇^γ denotes the fractional Riesz operator for $1 < \gamma < 2$ defined as

$$\begin{aligned} \nabla^\gamma u &= \frac{\partial^\gamma u}{\partial |x|^\gamma} + \frac{\partial^\gamma u}{\partial |y|^\gamma} \\ &= -\frac{1}{2\cos\left(\frac{\pi\gamma}{2}\right)}({}_{RL}D^\gamma_{-\infty,x}u + {}_{RL}D^\gamma_{x,+\infty}u) - \frac{1}{2\cos\left(\frac{\pi\gamma}{2}\right)}({}_{RL}D^\gamma_{-\infty,y}u + {}_{RL}D^\gamma_{y,+\infty}u). \end{aligned}$$

The operators ${}_{RL}D^\gamma_{-\infty,x}$ and ${}_{RL}D^\gamma_{x,+\infty}$ are defined as follows:

$${}_{RL}D^\gamma_{-\infty,x}u = \frac{\partial^2}{\Gamma(n-\gamma)\partial x^2}\int_{-\infty}^{x}(x-s)^{1-\gamma}u(s,y,t)ds,$$

$${}_{RL}D^\gamma_{x,+\infty}u = \frac{\partial^2}{\Gamma(n-\gamma)\partial x^2}\int_{-\infty}^{x}(s-x)^{1-\gamma}u(s,y,t)ds,$$

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where Γ is the gamma function. We will define the anomalously diffusive operator $(-\Delta)^{\frac{\gamma}{2}}$, for which

$$F\left(-(-\Delta)^{\frac{\gamma}{2}}u\right)(\xi) = -|\xi|^{\gamma}Fu(\xi),\ \xi \in \mathbb{R}^n,\ \gamma > 0.$$

For $0 < \gamma < 1$, it is commonly known as a subdiffusion; if $1 < \gamma < 2$, then according to [24], it is known as a superdiffusion and may be described using the Fourier transform. In the basic situation $\gamma = 2$, the operator $(-\Delta)^{\frac{\gamma}{2}}$ becomes the typical diffusion operator (Laplacian operator). For notational convenience, the anomalous diffusion operator $-(-\Delta)^{\frac{\gamma}{2}}$ is usually abbreviated as ∇^{γ}. Therefore we may write

$$F\left(\nabla^{\gamma}u\right)(\xi) = -|\xi|^{\gamma}Fu(\xi), \quad \xi \in \mathbb{R}^n, \quad \gamma > 0.$$

The operator $(-\Delta)^{s}_{\Omega}$ is the regional fractional Laplacian [25–27] defined by

$$(-\Delta)^{s}_{\Omega}u(x) = \lim_{\delta \to 0}(-\Delta)^{s}_{\Omega,\delta}u(x).$$

On the other hand, it cane be defined as

$$\begin{aligned}(-\Delta)^{s}_{\Omega}u(x) &= C_{n,s}PV\int_{\Omega}\frac{u(x)-v(y)}{|y-x|^{n+2S}}dy \\ &= C_{n,s}\lim_{\delta\to 0}\int_{\{y\in\Omega,|y-x|>\delta\}}\frac{u(x)-v(y)}{|y-x|^{N+2S}}dy, \quad \delta > 0,\end{aligned}$$

where PV denotes the Cauchy principal value, and the constant $C_{n,s}$ is specified by*colon*

$$C_{n,s} = \frac{2^{2s}\Gamma\left(\frac{n+2s}{2}\right)s}{\pi^{\frac{n}{2}}\Gamma(1-s)} \quad \text{for} \quad s \in \left]\tfrac{1}{2}, 1\right[.$$

After all these operations, the fractional Lengyel–Epstein fractional Laplacian model is as follows:

$$\begin{cases} \dfrac{du}{dt} = -(-\Delta)^{s}_{\Omega}u + a - u - \dfrac{4uv}{1+u^2}, \\ \dfrac{dv}{dt} = \sigma\left[-c(-\Delta)^{s}_{\Omega}v + b\left(u - \dfrac{uv}{1+u^2}\right)\right], \\ u(x,0) = u_0 > 0, \quad v(x,0) = v_0 > 0, \\ \partial_{\nu}u(x) = \partial_{\nu}v(x) = 0, \end{cases} \tag{1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with Lipschitz-continuous boundary $\partial\Omega$ and $u \in W^{s,2} \cap C^{s,2}\left(\overline{\Omega}\right)$ with $u \in C^{2,s}(\Omega)$, i.e.,

$$u \in C^2(\Omega) \quad \text{and} \quad \int_{\Omega}\int_{\Omega}\frac{|u(x)-v(y)|^2}{|y-x|^{n+2s}}dxdy < \infty.$$

The spectrum of the regional fractional Laplace operator $(-\Delta)^{s}_{\Omega}$ with a fractional Neumann boundary condition is discrete due to its compact resolvent, and the spectrum of $(-\Delta)^{s}_{\Omega}$ is an increasing sequence of real numbers $0 = \lambda^{N}_{0,s} \leq \lambda^{N}_{1,s} < \lambda^{N}_{2,s} \leq \cdots \leq \lambda^{N}_{n,s} \leq \cdots$ that

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converges to $+\infty$. Moreover, 0 is an eigenvalue of $(-\Delta)^s_\Omega$, where $\lambda^N_{n,s}$ is the nth eigenvalue of the fractional Laplace operator's realization in $L^2(\Omega)$ with fractional Neumann-type boundary conditions.

In recent years, a significant number of academics have developed concepts and carried out research associated with the fractional Laplacian and modeling a variety of physical phenomena. Some of the phenomena in this category are molecular dynamics, quasi-geostrophic flows, turbulence, and water waves. Additionally, this category encompasses the phenomenon of anomalous diffusion. We refer to [28, 29]. In addition to this, it has a wide range of applications in the disciplines of probability and finance [30–32]. Specifically, the fractional Laplacian may be regarded as the infinitesimal generator of a stable Lévy process [31]. This has been shown to be the reality. It is present in the propagation of flames and chemical reactions in liquids, population dynamics, geographical fluid dynamics, and American options in finance, as well as in anomalous diffusions in plasmas. For a comprehensive explanation of the function spaces involved in the study of the operator and a comprehensive list of references pertinent to the topic, we refer the readers to the survey work by Di Nezza, Palatucci, and Valdinoci [33].

The remaining sections of the paper are structured as follows. In Sect. 2, a unique bounded solution has been shown to exist. In Sect. 3, we study the fundamental properties of nonconstant positive solutions. In Sect. 4, we show that nonconstant positive solutions do not exist when the reactor is sufficiently big or when the diffusion coefficient is below a threshold that depends on the size of the reactor. In Sect. 5, we establish the conditions for Turing instability.

2 Invariant region

We will demonstrate in this section that system (1) admits the invariant region

$$\Re_a =]0, a[\times \left]0, 1 + a^2\right[. \tag{2}$$

Definition 1 A rectangle $\Re =]0, r_1[\times]0, r_2[$ is said to be an invariant rectangle if the vector field (f, g) on the boundary $\partial\Re$ points inside, i.e.,

$$\begin{cases} f(0, v) \geq 0, \quad f(r_1, v) \leq 0 \quad \text{for } 0 < v < r_2, \\ g(u, 0) \geq 0, \quad g(u, r_2) \leq 0 \quad \text{for } 0 < u < r_1. \end{cases} \tag{3}$$

We will use the following inequality [26, 34]:

$$\int_\Omega v(x)\,(-\Delta)^s_\Omega u(x)dx = \frac{C_{n,s}}{2}\int_\Omega\int_\Omega \frac{\left(u(x) - u(y)\right)\left(v(x) - v(y)\right)}{|y - x|^{n+2s}}dxdy, \tag{4}$$

$$\lambda^N_{1,s}\int_\Omega |u|^2 \leq \frac{C_{n,s}}{2}\int_\Omega\int_\Omega \frac{\left(u(x) - u(y)\right)^2}{|y - x|^{n+2s}}dxdy. \tag{5}$$

Proposition 1 *For all $x \in \Omega$ and $t > 0$, the initial-boundary value problem (1) has a unique solution (u, v). Additionally, there are two positive constants C_1 and C_2, which are entirely dependent on a, u_0, and v_0, such that*

$$C_1 < u(x, t), v(x, t) < C_2, \quad x \in \Omega, t > 0. \tag{6}$$

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Proof By [35, 36] the solutions of system (1) locally exist and are unique. We apply Theorem 6 to get the global existence and boundedness. We consider the rectangle

$$\Re =]U_1, U_2[\times]V_1, V_2[, \tag{7}$$

where

$$\begin{aligned} U_2 &= \max\left\{a, \max_{x\in\overline{\Omega}} u_0(x)\right\}, & V_2 &= \max\left\{2 + U_2^2, \max_{x\in\overline{\Omega}} v_0(x)\right\}, \\ U_1 &= \min\left\{\frac{a}{1+4V_2}, \min_{x\in\overline{\Omega}} u_0(x)\right\}, & V_1 &= \min\left\{\frac{1}{2}, \min_{x\in\overline{\Omega}} v_0(x)\right\}. \end{aligned}$$

Obviously, the rectangle encompasses the initial functions $u_0(x)$ and $v_0(x)$. Using a calculation similar to that of [11], we get that for system (1), the region $\Re$ is positively invariant. The solutions demonstrate their globality and uniqueness. Finally, we can now choose two constants

$$C_1 = \min\{U_1, V_1\} \quad \text{and} \quad C_2 = \max\{U_2, V_2\}. \qquad \square$$

Proposition 2 *Let $u(x,t)$ and $v(x,t)$ be the unique solution of (1). Then, for every $x \in \overline{\Omega}$,*

$$\limsup_{t\to\infty} u < a \quad \textit{and} \quad \limsup_{t\to\infty} v < 1 + a^2.$$

Proof According to Proposition 1, there exists $\varepsilon > 0$ such that

$$\varepsilon < \frac{4uv}{1+u^2}$$

for all $x \in \overline{\Omega}$ and $t > 0$. Let $\widetilde{u} = \widetilde{u}(t)$ be the unique solution to the following Cauchy problem:

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = \tilde{a} - \tilde{u}, \\ \tilde{u}(0) = 2\max_{x\in\overline{\Omega}} u_0(x), \end{cases} \tag{8}$$

with

$$\tilde{a} = a - \frac{\varepsilon}{2}$$

Let $\hat{u} = u - \tilde{u}$. By (1) and (8) we have that

$$\begin{cases} -\hat{u}_t - (-\Delta)^s_\Omega \hat{u} - \hat{u} = \dfrac{4uv}{1+u^2}\tilde{a} - a > 0, \\ \hat{u}(x,0) < 0, \end{cases} \tag{9}$$

which clearly yields $\hat{u}(x,t) < 0$ for all $t > 0$ and $x \in \overline{\Omega}$. The following argument is based on a contradiction. Suppose that this is false. Then there is $T > 0$ such that $\hat{u}(x,t) < 0$ for all $(x,t) \in \overline{\Omega} \times]0, T[$ and $\hat{u}(x,T) = 0$ for some $x \in \overline{\Omega}$. By continuity we have

$$\max_{x\in\overline{\Omega}} \hat{u}(x,T) = 0.$$

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Now, if there exists $x_1 \in \Omega$ such that $\hat{u}(x_1, T) = 0$, then

$$\hat{u}_t(x_1, T) \geq 0,$$

and

$$(-\Delta)^s_\Omega \hat{u}(x_1, T) \geq 0,$$

which leads to the conclusion that

$$-\hat{u}_t(x_1, T) - (-\Delta)^s_\Omega \hat{u}(x_1, T) - \hat{u}(x_1, T) \leq 0 \tag{10}$$

by (9) and (10), leading to a contradiction. If $\hat{u}(x, T) < 0$, then all $x \in \Omega$. After that, there would exist some $x_1 \in \partial\Omega$ such that $\hat{u}(x_1, T) = 0$. Then we have

$$-\hat{u}_t(x_1, T) - (-\Delta)^s_\Omega \hat{u}(x_1, T) - \hat{u}(x_1, T) > 0. \tag{11}$$

By continuity there exists a small ball Ω' in Ω with $\partial\Omega' \cap \Omega = \{x_1\}$ such that $\hat{u}(x, T) < 0$ for all $x \in \Omega'$. We can rewrite (9) as

$$-(-\Delta)^s_\Omega \hat{u} = \hat{u} + \frac{4uv}{1+u^2} + \tilde{a} - a > 0 \quad \text{in } \Omega' \times \{T\}.$$

We thus obtain $(-\Delta)^s_\Omega(-\hat{u}) > 0$ and $-\hat{u} > 0$ in $\Omega' \times \{T\}$. Then by the Hopf lemma [37] we have $\partial_\nu \hat{u}(x_1, T) < 0$, which contradicts (1). Thus

$$u(x,t) < \tilde{u}(t) \quad \textit{forall } t > 0 \text{ and } x \in \overline{\Omega}. \tag{12}$$

The comparison result for $v = v(x,t)$ and $\tilde{v} = \tilde{v}(t)$ is shown below. Assume that C_1 is the constant specified in Proposition 1 and $\varepsilon_0 > 0$. We define the function $\widetilde{v}(t)$ as the only solution to the initial value problem

$$\begin{cases} \dfrac{d\tilde{v}}{dt} = \sigma b \tilde{g}(\tilde{u}, \tilde{v}), \\ \tilde{v}(0) = 2\max_{x\in\overline{\Omega}} v_0(x), \end{cases} \tag{13}$$

where

$$\tilde{g}(\tilde{u}, \tilde{v}) = \sup_{C_1 < \varepsilon < \tilde{u}} \left(\varepsilon - \frac{\varepsilon(\tilde{v} - \varepsilon_0)}{1+\varepsilon^2} \right)$$

with

$$\tilde{a}^2 + \varepsilon_0 < a^2.$$

Let $\hat{v} = v - \tilde{v}$. Therefore $\hat{v}(x,t) < 0$ for all $x \in \overline{\Omega}$, which clearly results in $\hat{v}(x,t) < 0$ for all $t > 0$ and $x \in \overline{\Omega}$. In fact, we will continue with a contradiction argument. Suppose that this

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is false. Then there exists $T > 0$ such that $\hat{v}(x,t) < 0$ for all $(x,t) \in \overline{\Omega} \times]0, T[$ and $\hat{v}(x,t) = 0$ for some $x \in \overline{\Omega}$. By continuity we have

$$\max_{x \in \overline{\Omega}} \hat{v}(x,T) = 0.$$

Now, if there exists $x_1 \in \Omega$ such that $\hat{v}(x_1,T) = 0$, then

$$\hat{v}_t(x_1,T) \geq 0,$$

and

$$(-\Delta)^s_{\Omega} \hat{v}(x_1,T) \geq 0,$$

and hence

$$-\hat{v}_t(x_1,T) - (-\Delta)^s_{\Omega} \hat{v}(x_1,T) \leq 0.$$

In (1) and (13), on the other hand, we have

$$-\hat{v}_t - \sigma c (-\Delta)^s_{\Omega} \hat{v} = \sigma b \left[\tilde{g}(\tilde{u},\tilde{v}) - \left(u - \frac{uv}{1+u^2} \right) \right]. \tag{14}$$

Setting $\tilde{v} = v$ and $\tilde{u} > u$, as in the proof of [11, Proposition 2.2], we can see that

$$-\hat{v}_t - \sigma c (-\Delta)^s_{\Omega} \hat{v} > 0.$$

This results in a contradiction. If $\hat{v}(x,T) < 0$, then all $x \in \Omega$. After that, there would exist some $x_1 \in \partial\Omega$ such that $\hat{v}(x_1,T) = 0$. Then we have

$$-\hat{v}_t(x_1,T) - \sigma c (-\Delta)^s_{\Omega} \hat{v}(x_1,T) \geq 0.$$

By continuity there is a small ball Ω'' in B with $\partial\Omega'' \cap \Omega = \{x_1\}$ such that $\hat{v}(x,T) < 0$ for all $x \in \Omega''$. Then (9) can be rewritten as

$$-\hat{v}_t - \sigma c (-\Delta)^s_{\Omega} \hat{v} > 0 \quad \text{in } \Omega'' \times \{T\}.$$

Considering (14) in $\Omega'' \times \{T\}$, we have $(-\Delta)^s_{\Omega}(-\hat{v}) \geq 0$ and by Hopf lemma [37] that $\partial_v \hat{v}(x_1,T) > 0$, which contradicts (1), and therefore

$$v(x,t) < \tilde{v}(t) \quad \text{for all } t > 0, \quad x \in \overline{\Omega}. \tag{15}$$

Finally, we consider the following system:

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = \tilde{a} - \tilde{u}, \\ \dfrac{d\tilde{v}}{dt} = \sigma b \tilde{g}(\tilde{u},\tilde{v}), \end{cases}$$

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in $\Re$. Using the arguments in [11] and the definitions of $\tilde{u}$, $\tilde{v}$, and $\tilde{g}$, we obtain that $\lim_{t\to\infty}\tilde{u}(t) = \tilde{a}$ and $\lim_{t\to\infty}\tilde{v}(t) = 1 + \tilde{u}^2 + \varepsilon_0$.From (15) and (12) we have $\tilde{a} < a$ and $1 + \tilde{u}^2 + \varepsilon_0 < 1 + a^2$, and hence the proof is complete. □

3 Properties of nonconstant soluttions

In this section, we examine the basic properties of nonhomogeneous stable states of the Lengyel–Epstein system with regional fractional Laplacian, specifically positive nonconstant solutions to the system

$$\begin{cases} -(-\Delta)^s_\Omega u - a - u - \dfrac{4uv}{1+u^2} = 0, \\ -d(-\Delta)^s_\Omega v + u - \dfrac{uv}{1+u^2} = 0, \\ \partial_\nu u(x) = \partial_\nu v(x) = 0, \end{cases} \tag{16}$$

where $d = \dfrac{c}{b}$. Using Lemma 1 and the same technique as in proving Proposition 3.1 (see [11]), we get the following result.

Proposition 3 *Let* $(u, v) = (u(x), v(x))$ *be a positive solution of* (16). *Then*

$$\frac{a}{5+4a^2} < u < a \text{ and } 1 + \left(\frac{a}{5+4a^2}\right)^2 < v < l + a^2, \quad x \in \Omega. \tag{17}$$

For each positive solution $u = u(x)$, $v = v(x)$ of (16), let

$$\overline{u} = \frac{1}{|\Omega|} PV \int_\Omega u(x)\,dx \quad \text{and} \quad \overline{v} = \frac{1}{|\Omega|} PV \int_\Omega v(x)\,dx$$

be the averages of u and v over $\{y \in \Omega, |x-y| > \delta\}$, where $|\Omega|$ is the volume of Ω. Let $\phi = u - \overline{u}$ and $\psi = v - \overline{v}$. Then

$$\int_\Omega \phi dx = \int_\Omega \psi dx = 0. \tag{18}$$

Lemma 1 [35] *Let* $g \in C(\Omega \times \mathbb{R})$, *If* $w \in C(\Omega)$ *satisfies*

$$\begin{cases} (-\Delta)^s_\Omega w(x) + g(x, w(x)) \geq 0 \quad \text{in } \Omega, \\ w(x_0) = \min_{\overline{\Omega}} w, \end{cases} \tag{19}$$

then $g(x_0, w(x_0)) \geq 0$, *Alternatively, the opposite is also valid.*

Lemma 2 *Let u be a solution of* (16). *Then* $\overline{u} = \dfrac{a}{5}$.

Proof Write

$$w(x) = 4dv(x) - u(x). \tag{20}$$

From (16) we obtain

$$-(-\Delta)^s_\Omega w - a + 5u = 0. \tag{21}$$

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Integrating (21) over $\{y \in \Omega, |x-y| > \delta\}$, we obtain

$$\begin{aligned} PV \int_{\Omega} (5u-a)\,dx &= PV \int_{\Omega} (-\Delta)^{s}_{\Omega} w(y)\,dx \\ &= C_{n,s} PV \int_{\Omega} PV \int_{\Omega} \frac{u(x)}{|y-x|^{n+2s}} dxdy \\ &\quad - C_{n,s} PV \int_{\Omega} PV \int_{\Omega} \frac{u(y)}{|y-x|^{n+2s}} dydx. \end{aligned}$$

Because the roles of x and y in the previous integrals are symmetric, we have

$$PV \int_{\Omega} (5u-a)\,dx = 0,$$

and therefore $\overline{u} = \frac{a}{5}$. Let $\phi = u - \overline{u}$ and $\psi = v - \overline{v}$. Then

$$\int_{\Omega} \phi dx = \int_{\Omega} \psi dx = 0. \tag{22}$$

Lemma 3 *Let (u,v) be a nonconstant solution of* (16). *Then*

$$\textstyle\int_{\Omega} \phi\psi > 0, \quad \textit{and } \Upsilon > 0,$$

where

$$\Upsilon = \int_{\Omega}\int_{\Omega} \frac{(\phi(x)-\phi(y))(\psi(x)-\psi(y))}{|y-x|^{n+2s}} dxdy.$$

Proof Change (20) to

$$(-\Delta)^{s}_{\Omega} w = 5\phi.$$

Multiply this by $w = 4dv - u$ and integrate over Ω to obtain

$$\int_{\Omega} w(-\Delta)^{s}_{\Omega} w = 5\int_{\Omega} \phi w,$$

since $\phi = u - \overline{u}$, $\psi = v - \overline{v}$, $\int_{\Omega}\phi = \int_{\Omega}\psi = 0$, and (4). Thus we conclude

$$\int_{\Omega} \phi\psi = \frac{1}{20d}\left(\frac{C_{n,s}}{2}\int_{\Omega}\int_{\Omega} \frac{|(w(x)-w(y))|^2}{|y-x|^{n+2s}} + 5\int_{\Omega}\phi^2\right). \tag{23}$$

Multiplying (21) by ϕ and integrating over Ω, we obtain

$$5\textstyle\int_{\Omega}\phi^2 = 2dC_{n,s}\Upsilon - \frac{C_{n,s}}{2}\Phi, \quad \text{where} \quad \Phi = \int_{\Omega}\int_{\Omega} \frac{(\phi(x)-\phi(y))^2}{|y-x|^{N+2s}} dxdy,$$

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which means that

$$\Upsilon = \frac{1}{2dC_{n,s}}\left(\frac{C_{n,s}}{2}\Phi + 5\int_{\Omega}\phi^2\right). \tag{24}$$

From (23) and (24) the conclusion of this lemma immediately follows. Estimate (17) ensures the existence of a constant c_g dependent on a such that

$$\left|g\left(u,v\right)\right| = \left|u - \frac{uv}{1+u^2}\right| < c_g.$$

This completes the proof. □

Lemma 4 *There exists a constant* $C_g \equiv C_g\left(a, C_{ns}, \lambda_{1,s}^N, |\Omega|\right)$ *such that*

$$\int_{\Omega}\psi^2 + \Psi \leq C_g d^{-2}, \tag{25}$$

where

$$\Psi = \int_{\Omega}\int_{\Omega}\frac{\left(\psi\left(x\right) - \psi\left(y\right)\right)^2}{\left|y-x\right|^{n+2s}}dxdy.$$

Proof Using the second equation of system (16) along with the Schwarz inequality, we have

$$\begin{aligned} d\int_{\Omega}\psi\left(-\Delta\right)_{\Omega}^{s}v &= \int_{\Omega}\psi g\left(u,v\right) \leq \int_{\Omega}|\psi|\left|g\left(u,v\right)\right| \leq c_g\int_{\Omega}|\psi| \\ &\leq c_g\sqrt{|\Omega|}\left(\int_{\Omega}|\psi|^2\right)^{\frac{1}{2}}. \end{aligned}$$

We may obtain the following result: Using equation (5), we deduce the formula

$$d\frac{C_{n,s}}{2}\Psi \leq c_g\sqrt{|\Omega|}\left(\int_{\Omega}|\psi|^2\right)^{\frac{1}{2}}.$$

Moreover,

$$\int_{\Omega}|\psi|^2 \leq \frac{C_{n,s}}{2\lambda_{1,s}^N}\Psi$$

may be inferred from the Poincaré inequality. The outcome is

$$\Psi \leq \frac{2}{C_{n,s}}\frac{c_g^2|\Omega|}{\lambda_{1,s}^N}d^{-2}.$$

By employing equation (5) again we obtain equation (25) by utilizing the value

$$C_g = c_g^2|\Omega|\left(C_{n,s} + 2\lambda_{1,s}^N\right)/\left(\lambda_{1,s}^N\right)^2.$$

□

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Lemma 5 *Let* (u, v) *be a nonconstant solution to problem* (16). *Then*

$$\frac{3\left(\lambda_{1,s}^{N}\right)^{2}}{3\left(\lambda_{1,s}^{N}\right)^{2}+15\lambda_{1,s}^{N}+20}<\frac{\Phi}{d^{2}\Psi}<16. \tag{26}$$

Proof From (20) and (24) we obtain

$$\begin{aligned}\int_{\Omega}\int_{\Omega}\frac{\left(w(x)-w(y)\right)^{2}}{|y-x|^{n+2s}}dxdy &= 16d^{2}\Psi-8d\Upsilon+\Phi\\ &= 16d^{2}\Psi-\Phi-\frac{20}{C_{n+2s}}\int_{\Omega}\phi^{2},\end{aligned}$$

and, consequently,

$$\Phi+\frac{20}{C_{n,s}}\int_{\Omega}\phi^{2}\leq 16d^{2}\Psi. \tag{27}$$

Therefore the second inequality of (26) follows. We start the proof by claiming

$$16d^{2}\Psi=\Phi+\frac{10}{C_{n,s}}\int_{\Omega}\phi^{2}+\frac{40d}{C_{n,s}}\int_{\Omega}\phi\psi. \tag{28}$$

Continuing the above computation, we get that

$$16d^{2}\Psi=\int_{\Omega}\int_{\Omega}\frac{\left(w(x)-w(y)\right)^{2}}{|y-x|^{n+2s}}dxdy+\Phi+\frac{20}{C_{n,s}}\int_{\Omega}\phi^{2}.$$

Using (23), we have (28). Next, we recall the elementary Cauchy inequality: for all real numbers ξ, ζ, and $\kappa>0$,

$$\xi\zeta\leq\frac{1}{4\kappa}\xi^{2}+\kappa\zeta^{2}. \tag{29}$$

Now from (28) and (29) it follows that

$$\begin{aligned}16d^{2}\Psi &\leq \Phi+\frac{10}{C_{n,s}}\int_{\Omega}\phi^{2}+\frac{40}{3\lambda_{1,s}^{N}C_{n,s}}\int_{\Omega}\phi^{2}+\frac{30d^{2}\lambda_{1,s}^{N}}{C_{n,s}}\int_{\Omega}\psi^{2}\\ &\leq\left[1+\frac{20}{3\left(\lambda_{1,s}^{N}\right)^{2}}+\frac{5}{\lambda_{1,s}^{N}}\right]\Phi+15d^{2}\Psi.\end{aligned}$$

Consequently,

$$d^{2}\Psi\leq\left[1+\frac{20}{3\left(\lambda_{1,s}^{N}\right)^{2}}+\frac{5}{\lambda_{1,s}^{N}}\right]\Phi, \tag{30}$$

demonstrating the first inequality of (26). □

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Lemma 6 *Let* (u, v) *be a nonconstant solution to system* (16). *Then*

$$\frac{6\left(\lambda_{1,s}^{N}\right)^{3}}{\left(2\lambda_{1,s}^{N}+C_{n,s}\right)\left(3\left(\lambda_{1,s}^{N}\right)^{2}+15\lambda_{1,s}^{N}+20\right)}<\frac{\Phi+\frac{1}{C_{n,s}}\int_{\Omega}\phi^{2}}{d^{2}\left(\Psi+\int_{\Omega}\psi^{2}\right)}<16. \tag{31}$$

Proof Using (5), we have

$$\Psi+\int_{\Omega}\psi^{2}\leq\frac{2\lambda_{1,s}^{N}+C_{n,s}}{2\lambda_{1,s}^{N}}\Psi.$$

This and the first inequality of (26) imply the first inequality of (31). The second inequality of (31) is determined by (27). □

4 Nonexistence of nonconstant positive solutions

In this section, we discuss nonconstant positive solutions to (16). Our theorems indicate that the reactor size and diffusion coefficients prevent nonconstant positive solutions.

Theorem 4 *There exists a constant* $d_0=d_0\left(a,C_{n,s},\lambda_{1,s}^{N}\right)>0$ *such that problem* (16) *has no nonconstant solution for* $0<d<d_0$.

Proof Integrating the second equation of system (16) over the region Ω and multiplying it by the variable ψ result in the formula

$$\begin{aligned}d\frac{C_{n,s}}{2}\Psi&=\int_{\Omega}\phi\psi-\int_{\Omega}\frac{uv\psi}{1+u^{2}}\\&=\int_{\Omega}\phi\psi-\int_{\Omega}\frac{u\psi^{2}}{1+u^{2}}-\int_{\Omega}\left(\frac{1-u\overline{u}}{\left(1+u^{2}\right)\left(1+\overline{u}^{2}\right)}\right)\overline{v}\phi\psi.\end{aligned}$$

The a priori estimates in Proposition 3 imply that

$$d\frac{C_{n,s}}{2}\Psi\leq C_{1}(a)\int_{\Omega}\phi\psi-C_{2}(a)\int_{\Omega}\psi^{2}.$$

Employing the Cauchy inequality and inequality (5), we obtain

$$\Psi\leq\frac{\left(C_{1}(a)\right)^{2}C_{n,s}}{8dC_{2}(a)\lambda_{1,s}^{N}}\Phi\leq\frac{d}{d_{0}}\Psi, \tag{32}$$

which is a result of equation (26), where

$$d_{0}=d_{0}\left(a,C_{n,s},\lambda_{1,s}^{N}\right)=\frac{C_{2}(a)\lambda_{1,s}^{N}}{2\left(C_{1}(a)\right)^{2}C_{n,s}}. \tag{33}$$

Clearly, if $d<d_0$, then $\Psi=0$, and hence $\Phi=0$ by (26) once again. As a result, φ and ψ are constants. □

Theorem 5 *There exists a positive constant* $\Lambda=\Lambda(a)>0$ *such that if* $\lambda_{1,s}^{N}>\Lambda$, *then problem* (16) *has no nonconstant solutions*.

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Proof Integrating the first equation of system (16) over the region Ω and multiplying it by the variable ϕ yield

$$\frac{C_{n,s}}{2}\Phi = -\int_{\Omega}\phi^2 - 4\left[\int_{\Omega}\frac{v\phi^2}{1+u^2}+\int_{\Omega}\frac{\overline{u}\phi\psi}{1+u^2}-\int_{\Omega}\frac{\overline{uv}\,(\overline{u}+u)\,\phi^2}{\left(1+u^2\right)\left(1+\overline{u}^2\right)}\right] \tag{34}$$

$$\leq 4\int_{\Omega}\frac{\overline{uv}\,(\overline{u}+a)}{\left(1+\overline{u}^2\right)}\phi^2 - 4\int_{\Omega}\frac{\overline{u}}{1+u^2}\phi\psi. \tag{35}$$

Using the a priori estimates in Proposition 3, we obtain the estimate

$$\frac{C_{n,s}}{2}\Phi \leq C_3\int_{\Omega}\phi^2 + C_3\int_{\Omega}\phi\psi, \quad \text{where} \quad C_3 = \max\left(4\frac{\overline{uv}\,(\overline{u}+a)}{\left(1+\overline{u}^2\right)}, \frac{4\left(5+4a^2\right)^2\overline{u}}{\left(5+4a^2\right)^2+a^2}\right). \tag{36}$$

We can obtain

$$\int_{\Omega}\phi\psi \leq \left(\int_{\Omega}|\phi|^2\right)^{\frac{1}{2}}\left(\int_{\Omega}|\psi|^2\right)^{\frac{1}{2}} \leq \frac{C_{n,s}}{2\lambda_{1,s}^N}\Phi^{\frac{1}{2}}\Psi^{\frac{1}{2}}$$

$$\leq C_4\,\frac{C_{n,s}}{2}d^{-\frac{1}{2}}\left(\lambda_{1,s}^N\right)^{-\frac{3}{2}}\Phi, \quad \text{where} \quad C_4 = \left(\frac{\left(C_1(a)\right)^2 C_{N,s}}{8C_2(a)}\right)^{\frac{1}{2}},$$

by combining the Cauchy inequality with inequality (5), as demonstrated in (32). This and (36) lead us to the conclusion that

$$\Phi \leq \frac{C}{\lambda_{1,s}^N}\left(1+\left(d\lambda_{1,s}^N\right)^{-\frac{1}{2}}\right)\Phi, \quad \text{where} \quad C = \max\left(C_3, C_3\times C_4\right). \tag{37}$$

Now, if $d \geq 1$, then there exists $\Lambda > 0$ such that once $\lambda_{1,s}^N > \Lambda$ has been selected, we have

$$\frac{C}{\lambda_{1,s}^N}\left(1+\left(d\lambda_{1,s}^N\right)^{-\frac{1}{2}}\right) \leq \frac{C}{\lambda_{1,s}^N}\left(1+\left(\lambda_{1,s}^N\right)^{-\frac{1}{2}}\right) < 1.$$

Consequently, the inequality in (37) gives $\Phi = 0$. The estimates presented in (26) also imply $\Psi = 0$ and the nonexistence of nonconstant positive solutions. Alternatively, if $d < 1$, then we can choose $\lambda_{1,s}^N$, which, according to Theorem 4, results in the nonexistence. □

5 Turing instability

This section primarily addresses the stability of steady-state constant solutions. System (1) has a distinct steady-state solution $(u^*, v^*) = \left(\alpha, 1+\alpha^2\right)$, where $\alpha = \frac{a}{5}$. In the absence of diffusion, system (1) is reduced to

$$\begin{cases} u_t = a - u - \frac{4uv}{1+u^2} = f(u,v), \\ v_t = -(-\Delta)_{\Omega}^{s} v + g(u,v) = g(u,v). \end{cases} \tag{38}$$

The Jacobian matrix for (38) evaluated at (u^*, v^*) is

$$J\left(u^*, v^*\right) = \begin{pmatrix} f_0 & f_1 \\ \sigma b g_0 & \sigma b g_1 \end{pmatrix} = \begin{pmatrix} \frac{3\alpha^2-5}{1+\alpha^2} & -\frac{4\alpha}{1+\alpha^2} \\ \sigma b\frac{2\alpha^2}{1+\alpha^2} & -\sigma b\frac{\alpha}{1+\alpha^2} \end{pmatrix}. \tag{39}$$

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Suppose that

$$3\alpha^2 - 5 > 0 \tag{40}$$

is a situation where system (1) functions as an activator–inhibitor system. The equilibrium of the ordinary differential equation (ODE) system (38) is considered asymptotically stable if the condition

$$3\alpha^2 - 5 < \sigma b\alpha \tag{41}$$

is satisfied, as stated by [11]. The condition

$$0 < 3\alpha^2 - 5 < \sigma b\alpha \tag{42}$$

may be deduced by combining equations (40) and (41). Model (1) establishes a stable activator–inhibitor system in the absence of diffusion. The eigenvalue problem in the fractional Laplacian operator $(-\Delta)^s_\Omega$ is represented as

$$\begin{cases} (-\Delta)^s_\Omega = \lambda^N_{i,s}\phi_{i,j}, & x \in \Omega, \\ \dfrac{\partial \phi_{i,j}}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

where each $\lambda^N_{i,s}$ has an algebraic multiplicity of $m_i \geq 1$. The normalized eigenfunctions $\{\phi_{i,j}\}, i \geq 0, 1 \leq j \leq m_i$, represent the orthonormal basis in $L^2(\Omega)$. These eigenfunctions are connected with $\lambda^N_{1,s}$. Given the condition

$$\lambda^N_{1,s} < f_0 = \frac{3\alpha^2 - 5}{1 + \alpha^2}, \tag{43}$$

let $i_\alpha = i_\alpha(\alpha, \Omega)$ be the largest positive integer satisfying

$$\lambda^N_{i,s} < f_0 \quad \text{for} \quad i \leq i_\alpha.$$

For this specific case, we will analyze the value of

$$\tilde{d} = \tilde{d}(\alpha, \Omega) = \min_{1 \leq i \leq i_\alpha} d_i, \quad d_i = \frac{\alpha}{1 + \alpha^2} \frac{\lambda^N_{1,s} + 5}{\lambda^N_{i,s}\left(f_0 - \lambda^N_{i,s}\right)}. \tag{44}$$

Lemma 7 *Assume that (H) holds. If* $\lambda^N_{1,s} \geq f_0$ *or* $\lambda^N_{1,s} < f_0$ *and* $0 < d = \dfrac{c}{b} < \tilde{d}$, *then the constant steady state* (u^*, v^*) *is asymptotically stable. If* $\lambda^N_{1,s} < f_0$ *and* $d > \tilde{d}$, *then* (u^*, v^*) *is unstable.*

Proof We analyze a disturbance expressed in the form [23]

$$\frac{\partial U}{\partial t} = \begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix} = \begin{pmatrix} u(t) - u^* \\ v(t) - v^* \end{pmatrix}.$$

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For convenience, we will continue to use $u(t)$ and $v(t)$ to refer to $\widetilde{u}(t)$ and $\widetilde{v}(t)$. The linearized system (1) is defined by the equation

$$\frac{\partial U}{\partial t} = DA^s U + JU$$

with

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & \sigma c \end{pmatrix}, \quad A^s U = \begin{pmatrix} -(-\Delta)^s_\Omega & 0 \\ 0 & -(-\Delta)^s_\Omega \end{pmatrix}.$$

The eigenvalue issue is expressed by the equation

$$\left(DA^s + J\right)(U)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

where μ is an eigenvalue of $DA^s + J(u^*, v^*)$ if and only if μ is an eigenvalue of the matrix $-\lambda^N_{i,s} D + J$ for each $i \geq 0$. The characteristic polynomial

$$\det\left(\mu I - \lambda^N_{i,s} D - J\left(u^*, v^*\right)\right) = \mu^2 + P_i \mu + \sigma b Q_i$$

is therefore necessary to analyze the local stability at (u^*, v^*). It may also be stated as

$$\sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} \begin{pmatrix} f_0 - \lambda^N_{i,s} - \mu & f_1 \\ \sigma b g_0 & \sigma b g_1 - \sigma c \lambda^N_{i,s} - \mu \end{pmatrix} \begin{pmatrix} a_{i,j} \\ b_{i,j} \end{pmatrix} \phi_{i,j} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\phi = \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} a_{i,j} \phi_{i,j} \quad \text{and} \quad \psi = \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} b_{i,j} \phi_{i,j}.$$

Condition (42) yields

$$P_i = (1 + \sigma c)\lambda^N_{i,s} - \left(\frac{3\alpha^2 - 5}{1 + \alpha^2} - \frac{\sigma b \alpha}{1 + \alpha^2}\right) > 0 \quad \text{and} \quad Q_i = d\lambda_i\left(\lambda^N_{i,s} - f_0\right) + \frac{\alpha\left(5 + \lambda^N_{i,s}\right)}{1 + \alpha^2}, \tag{45}$$

since $c = bd$. Obviously, $Q_0 > 0$ for $\lambda_0 = 0$. Now, if $\lambda^N_{1,s} \geq f_0$, then by (45) $Q_i > 0$ for all $i \geq 1$, so it follows that Re $\mu < 0$ for all eigenvalues μ, and the steady-state (u^*, v^*) is therefore asymptotically stable. If $\lambda^N_{1,s} < f_0$, then we have two cases.

- If $0 < d < \tilde{d}$, then

  $$\lambda^N_{i,s} < f_0, \quad \text{and} \quad d < d_i, \quad i \in [1, i_\alpha].$$

  Consequently, $Q_i > 0$ for every $i \in [1, i_\alpha]$. In addition, if $i > i_\alpha$, then $\lambda^N_{i,s} \geq f_0$ and $Q_i > 0$ by (45). The reasoning once again leads to the asymptotic stability of (u^*, v^*).
- If $d > \tilde{d}$, then we may infer that $k \in [1, i_\alpha]$ attains the minimum in (44). Thus $d > d_k$, which implies $Q_k < 0$, and thus (u^*, v^*) is unstable. □

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6 Conclusions

This work models the chlorite-iodide-malonic acid (CIMA) chemical reaction using a space-fractional version of the Lengyel–Epstein system. The primary objective was to ensure the presence of invariant regions, whereas this study further sought to demonstrate the existence, uniqueness, and positivity of global solutions. Furthermore, we analyzed the boundary value problem associated with the fractional elliptic Laplacian. From the results of our analysis we derived our first estimations of the essential characteristics of nonconstant steady-state solutions. This has allowed us to establish better results and the conditions necessary for the nonexistence of nonconstant positive solutions, which are contingent upon the sufficiently large reactor size, together with the diffusion coefficient. Finally, conditions ensuring the Turing instability of the system have been established using the characteristic polynomial and the eigenvalue problem associated with the fractional Laplacian operator. Thus, under these circumstances, the system allowed diffusion-driven instability, resulting in the formation of patterns. We defined the method to identify a set S under the initial and boundary conditions of a solution u of the system. Future studies will primarily concentrate on the existence of a nonconstant positive solution and the study of the asymptotic stability of the global system. Subsequently, we are going to study the global and local bifurcation structure of the nonconstant steady states.

Appendix

Suppose that if

$$\frac{\partial u_i}{\partial t} + d\,(-\Delta)^{s}_{\Omega}\, u_i = f_i\,(u) \quad \text{for } 1 \le i \le m, \tag{A.1}$$

where $u = (u_1, \ldots, u_i, \ldots, u_m)^T$, $d = (d_1, \ldots, d_i, \ldots, d_m)^T$, and $F = \left(f_1, \ldots, f_i, \ldots, f_m\right)^T$ are such that $f_i \colon \mathbb{R}^m \to \mathbb{R}^m$ lie in S, then u values also lie in S. Such a set S is known as an invariant set. Let D be a domain in $\mathbb{R}^m$ with closure $\overline{D}$. If S is a closed convex set in $\mathbb{R}^m$, and let S_ρ ($\rho > 0$) denote the parallel set of all points in $\mathbb{R}^m$ whose Euclidean distance to S is no greater than ρ (see [38]). The subsequent lemma and theorem are developed from lemma and Theorem 1 in [38].

Lemma 8 *Let S be a convex closed subset of $\mathbb{R}^m$. Suppose there exists $\varepsilon > 0$ such that for any $\rho \in]0, \infty[$, the parallel set S_ρ possesses the feature that if p is the outward normal at a position u^* on the border of S_ρ, then*

$$p \cdot F\left(u^*\right) < 0 \tag{A.2}$$

for all (x,t) in $D \times]0,T]$. If u is any solution of system (A.1) and all limit points of $u\,(x,t)$ as (x,t) approaches $\overline{D} \times \{0\}$ or $\partial D \times [0,T]$ lie in S, then $u\,(x,t) \in S$ in $D \times]0,T]$.

Proof Assume that for some $(x,t) \in D \times]0,T]$, $u\,(x,t)$ is not in S. Consequently, $u\,(x,t)$ exists outside of S_ρ for some $\rho \in]0,\varepsilon[$. A point $(x^*,t^*) \in D \times]0,T]$ exists due to continuity in such a manner that $u\,(x,t) \in S_\rho$ for $(x,t) \in D \times]0,t^*]$, where $u^* = u\,(x^*,t^*) \in \partial S_\rho$. Let p be the normal outside ∂S_ρ at u^*. Consequently, by hypothesis, $p \cdot F\,(u^*) < 0$. Alternatively, since S_ρ is convex, the function $p \cdot u$ has a maximum value in $D \times]0,t^*]$. Therefore, at

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position (x^*, t^*),

$$p \cdot \frac{\partial u(x,t)}{\partial t} \geq 0,$$

and

$$\langle p, u^* - u(x,t) \rangle \geq 0.$$

Integrating over $\{x \in \Omega, |x^* - x| > \delta\}$ such that $\delta > 0$ gives

$$\langle p, (-\Delta)^s_\Omega u^* \rangle \geq 0.$$

Therefore it follows from (A.1) that $p \cdot F(u) \geq 0$, a contradiction to (A.2). The conclusion that $u(x,t)$ cannot lie outside S for $(x,t) \in D \times]0,T]$ proves the lemma. □

Making certain smoothness assumptions about system (A.2) and the domain D, we may substitute criteria (A.2) on the borders of an entire family of parallel sets with the weaker condition

$$p \cdot F(u^*) \leq 0 \quad \text{for } (x,t) \text{ in } D \times]0,T], \tag{A.3}$$

Theorem 6 *Let S be a convex closed subset of $\mathbb{R}^m$ such that inequality (A.3) holds for every outward normal p at any boundary point u^* of S. Assume that F is a Lipschitz-continuous function on bounded sets. If $u(x,t)$ is a solution of (A.1) in $D \times]0,T]$, $u \in C^{2,s}(\Omega)$, and if the values of u on $\overline{D} \times \{0\}$ and on $\partial D \times [0,T]$ lie in S, then $u(x,t) \in S$ in $D \times]0,T]$.*

Proof For any point w of $\mathbb{R}^m$ outside S, there exists a unique point $q(w)$ on ∂S that is closest to w; $w - q(w)$ is, of course, an outer normal to ∂S at $q(w)$. We define $G = (g_1, \ldots, g_i, \ldots, g_m)^T$ such that $g_i \colon \mathbb{R}^m \to \mathbb{R}^m$ with

$$G(w) = \begin{cases} F(w) & \text{if } w \in S, \\ F(q(w)) - w + q(w) & \text{if } w \notin S. \end{cases}$$

For $1 \leq i \leq m$, the function g_i satisfies hypothesis (H_1). The problem

$$\begin{cases} \dfrac{\partial w}{\partial t} + (-\Delta)^s_\Omega w = G(w,x,t), \\ w(x,0) = u(x,0), \\ w(x,t)|_{\partial D} = u(x,t)|_{\partial D} \end{cases}$$

has a unique solution according to the hypothesis; $p = w^* - q(w^*)$ is an outward normal vector to ∂S_ρ at w^* and to ∂S at $q(w^*)$ if w^* is a border point of the parallel set S_ρ. Hence

$$p \cdot G(w^*) = (w^* - q(w^*)) \cdot F(u^*) - |w^* - q(w^*)|^2 \leq -\rho^2$$

by (A.3). The parallel sets of S therefore satisfy (A.2) with respect to the vector field G. Lemma 8 demonstrates that $w(x,t) \in S$ in $D \times]0,T]$. □

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Author contributions
Salim.Zidi. wrote the main manuscript text. reviewed the manuscript.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
The conducted research is not related to either human or animal use.

Competing interests
The authors declare no competing interests.

Received: 26 August 2024 Accepted: 2 November 2024 Published online: 18 December 2024

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